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Elastic wave-motion across a vertical discontinuity

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SUMMARY
Exact solutions are obtained for the displacement field in an elastic half-space composed of two quarter spaces welded together. The configuration is excited by a plane SH wave impinging upon the discontinuity at an arbitrary angle. The application of the Kontorovich–Lebedev transform to this boundary value problem leads to two simultaneous integral equations which are solved exactly. It is shown that the discontinuity may enhance the spectral displacements up to a factor of two. The results could be applied to propagation of seismic shear waves past fault zones in the earth's crust.

1. Introduction

In recent years considerable interest has been shown by many investigators to the propagation of elastic waves in parallel layered media [1]. The techniques developed for such configurations can model only vertical heterogeneities while solutions to problem involving lateral structural variations in elastic media are almost unknown. The reason is that the boundary conditions equations in layered media are reduced to a set of algebraic equations while the other types of non-homogeneity involve *ab initio* integral equations for the reflection coefficients. Certain approximations are sometimes helpful, *e.g.* [2] but useful working tools are still lacking in as much as the ensuing integral equations are not text-book items.

The interaction of elastic waves with dipping discontinuities is a problem which belongs to the above category. We have found however that for a vertical dip angle a solution is attainable in a closed form. It is hoped that these results could be generalized to more complicated physical regimes.

2. Basic equations, solutions and source geometry

Consider the wave equation for an homogeneous elastic solid [3]

$$\frac{\partial^2 \boldsymbol{D}}{\partial t^2} = v_p^2 \text{ grad div } \boldsymbol{D} - v_s^2 \text{ curl curl } \boldsymbol{D}\,, \tag{2.1}$$

where $\boldsymbol{D}$ is the displacement vector, t is the time and v_p and v_s are the longitudinal and shear wave velocities in the medium, respectively. If a primary excitation is generated by a line source parallel to the z-axis, the resulting field is independent of the z-coordinate. The general solution of Eqn. (2.1) is composed of three basic constituents. The first corresponds to horizontally polarized shear motion (known as SH waves)

$$\boldsymbol{D}_1 = \boldsymbol{M} = \boldsymbol{e}_z U(r, \theta) \mathrm{e}^{\mathrm{i}\omega t}\,, \quad \text{div } \boldsymbol{D}_1 = 0\,, \tag{2.2}$$

where (r, θ) are cylindrical polar coordinates in a vertical plane and $\boldsymbol{e}_z$ is a unit vector in the z-direction. Equation (2.1) is then reduced to

$$\nabla^2 U - \beta^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} - \beta^2 U = 0\,, \quad \beta = \mathrm{i}\,\frac{\omega}{v_s}\,. \tag{2.3}$$

The remaining solutions are used to describe the coupled compressional-shear wave motion (known as P-SV waves)

$$D_2 = L = \operatorname{grad} \psi \, e^{i\omega t}, \qquad \operatorname{curl} D_2 = 0, \tag{2.4}$$

$$D_3 = N = \operatorname{curl}(e_z \chi) e^{i\omega t}, \quad \operatorname{div} D_3 = 0, \tag{2.5}$$

with the corresponding equations

$$\nabla^2 \psi - \alpha^2 \psi = 0, \quad \nabla^2 \chi - \beta^2 \chi = 0, \quad \alpha = i \frac{\omega}{v_p}. \tag{2.6}$$

A torque line source generates waves of type M only while a compressional line source will produce both L and N solutions in the presence of a discontinuity. We shall treat only solutions of the M type.

A solution of Eqn. (2.3) which represents divergent waves is given by

$$U = \begin{Bmatrix} \operatorname{sh} m\theta \\ \operatorname{ch} m\theta \end{Bmatrix} K_{im}(\beta r). \tag{2.7}$$

These solutions are sufficient to describe the total field because reflections from discontinuities in the medium can be considered as waves diverging from a proper set of images.

The geometry of the problem is sketched in Fig. 1. An elastic half-space is composed of two wedges, welded together at $\theta = \alpha$. The constants of the right wedge are denoted by the subscript 1 and those of the left wedge by the subscript 2.

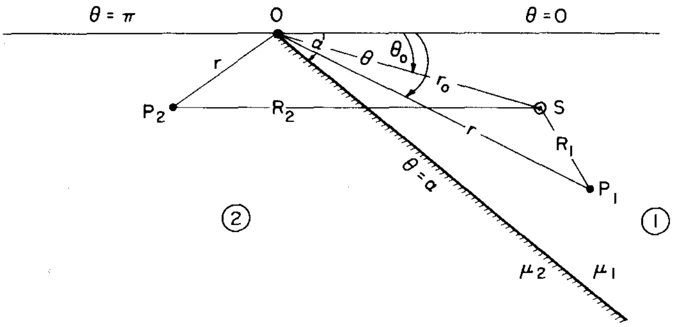


Figure 1. Geometry of line source at finite distance from the vertex.

A harmonic line source is placed at (r_0, θ_0) perpendicular to the plane of the paper, parallel to the z-axis at 0. The Green's function for the two-dimensional wave equation is $(2\pi)^{-1} K_0(\beta R)$ where $R = [r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)]^{\frac{1}{2}}$ is the observer-source distance. Using the integral form of the addition theorem for the Bessel functions, we represent the source in the form [4, p. 3]

$$U_0 = \frac{s_0}{2\pi} K_0(\beta R) = \frac{s_0}{2\pi} \int_0^\infty K_{im}(\beta r) K_{im}(\beta r_0) \operatorname{ch} m(\pi - |\theta - \theta_0|) dm, \tag{2.8}$$

$$0 \leqq |\theta - \theta_0| \leqq 2\pi,$$

where s_0 is the line-source constant, with the dimensions of spectral displacement. Note that for real values of m the function $K_{im}(\beta r)$ is real for real values of its argument. In Eqn. (2.8) it is understood that the function $\operatorname{ch} m(\pi + \theta - \theta_0)$ is used for $\theta < \theta_0$ and $\operatorname{ch} m(\pi - \theta + \theta_0)$ for $\theta > \theta_0$.

3. Integral equations for the reflection coefficients

Boundary conditions are as follows: The stress component in the z-direction must vanish over the entire free surface ($\theta = 0, \pi$). [The other two components are identically equal to zero for horizontally polarized shear (SH) waves]. At the common boundary of the two media ($\theta = \alpha$) we require a continuity of both the displacements and the stresses.

Assuming that the displacement in either wedge can be represented by an integral whose integrand is a linear combination of azimuthal outgoing and incoming waves with certain reflection coefficients, we have

$$U_1(r,\theta)=\frac{2s_0}{\pi^2}\int_0^\infty [A(m)\,\mathrm{sh}\,m\theta+B(m)\,\mathrm{ch}\,m\theta+\tfrac{1}{2}\,\mathrm{ch}\,m(\pi-|\theta-\theta_0|)]K_{im}(\beta_1 r)K_{im}(\beta_1 r_0)dm\,,\qquad 0\leqq\theta\leqq\alpha\,,\quad(3.1)$$

$$U_2(r,\theta)=\frac{2s_0}{\pi^2}\int_0^\infty [C(m)\,\mathrm{sh}\,m\theta+D(m)\,\mathrm{ch}\,m\theta]K_{im}(\beta_2 r)K_{im}(\beta_1 r_0)dm\,,\qquad \alpha\leqq\theta\leqq\pi\,.\quad(3.2)$$

Applying the boundary conditions

$$P_{z\theta}=\frac{1}{r}\frac{\partial U}{\partial\theta}=0 \text{ at } \theta=0 \text{ and } \theta=\pi\,,\tag{3.3}$$

$$P_{z\theta}(\theta=\alpha+0)=P_{z\theta}(\theta=\alpha-0)\,,\quad U_1(\theta=\alpha)=U_2(\theta=\alpha)\,,$$

we obtain two simultaneous integral equations for the unknown functions $B=X(m)$ and $D=Y(m)\,\mathrm{ch}\,\pi m$. Thus

$$A(m)=-\tfrac{1}{2}\,\mathrm{sh}\,m(\pi-\theta_0)\,,\quad C(m)=-Y(m)\,\mathrm{sh}\,\pi m\,,\quad \lambda=\frac{\mu_2}{\mu_1}\,,\tag{3.4}$$

$$\int_0^\infty X(m)\,\mathrm{ch}\,m\alpha\,K_{im}(\beta_1 r)K_{im}(\beta_1 r_0)dm-\int_0^\infty Y(m)\,\mathrm{ch}\,m(\pi-\alpha)K_{im}(\beta_2 r)K_{im}(\beta_1 r_0)dm=F(m,r)\,,\tag{3.5}$$

$$\int_0^\infty X(m)\,\mathrm{sh}\,m\alpha K_{im}(\beta_1 r)K_{im}(\beta_1 r_0)m\,dm+\lambda\int_0^\infty Y(m)\,\mathrm{sh}\,m(\pi-\alpha)K_{im}(\beta_2 r)K_{im}(\beta_1 r_0)m\,dm=G(m,r)\,.\tag{3.6}$$

The source terms on the right-hand side of Eqns. (6.10) and (6.11) are given by the integral expressions

$$F(m,r)=\tfrac{1}{2}\int_0^\infty [\mathrm{sh}\,m(\pi-\theta_0)\,\mathrm{sh}\,m\alpha-\mathrm{ch}\,m(\pi+\theta_0-\alpha)]K_{im}(\beta_1 r)K_{im}(\beta_1 r_0)dm\,,\tag{3.7}$$

$$G(m,r)=\tfrac{1}{2}\int_0^\infty [\mathrm{sh}\,m(\pi-\theta_0)\,\mathrm{ch}\,m\alpha+\mathrm{sh}\,m(\pi+\theta_0-\alpha)]K_{im}(\beta_1 r)K_{im}(\beta_1 r_0)m\,dm\,.\tag{3.8}$$

Multiplication of Eqns. (3.5) and (3.6) by $K_{i\tau}(\beta_1 r)r^{-1}dr$ followed by integration over $(0,\infty)$, yields, using (A.6)

$$\frac{\pi^2}{2\tau\,\mathrm{sh}\,\pi\tau}X(\tau)\,\mathrm{ch}\,\alpha\tau\,K_{i\tau}(\beta_1 r_0)-\int_0^\infty Y(m)Q(\tau,m)\,\mathrm{ch}\,m(\pi-\alpha)K_{im}(\beta_1 r_0)dm=\int_0^\infty F(m,r)K_{i\tau}(\beta_1 r)\frac{dr}{r}\,,\tag{3.9}$$

$$\frac{\pi^2\tau}{2\tau\,\mathrm{sh}\,\pi\tau}X(\tau)\,\mathrm{sh}\,\alpha\tau\,K_{i\tau}(\beta_1 r_0)+\lambda\int_0^\infty Y(m)Q(\tau,m)\,\mathrm{sh}\,m(\pi-\alpha)K_{im}(\beta_1 r_0)m\,dm=\int_0^\infty G(m,r)K_{i\tau}(\beta_1 r)\frac{dr}{r}\,,$$

where

$$Q(\tau, m) = \int_0^\infty K_{i\tau}(\beta_1 r) K_{im}(\beta_2 r) \frac{dr}{r}. \tag{3.10}$$

Elimination of $X(\tau)$ from Eqn. (3.9) leads to a single integral equation in the unknown transmission coefficient $Y(m, \theta_0, r_0)$:

$$\int_0^\infty Y(m) Q(\tau, m) K_{im}(\beta_1 r_0) \{\lambda m \text{ ch } \alpha\tau \text{ sh } m(\pi-\alpha) + \tau \text{ sh } \alpha\tau \text{ ch } m(\pi-\alpha)\} dm$$
$$= \tfrac{1}{2}\pi^2 \text{ ch } \tau\theta_0 K_{i\tau}(\beta_1 r_0). \tag{3.11}$$

Likewise, the integral equation for the second unknown function is

$$\int_0^\infty X_1(m) Q(m, \tau) K_{im}(\beta_1 r_0) \{\lambda\tau \text{ sh } \tau(\pi-\alpha) \text{ ch } m\alpha + m \text{ ch } \tau(\pi-\alpha) \text{ sh } m\alpha\} dm$$
$$= \int_0^\infty Q(m, \tau) K_{im}(\beta_1 r_0) \text{ sh } \pi m [\lambda\tau \text{ sh } \tau(\pi-\alpha) \text{ sh } m(\alpha-\theta_0)$$
$$+ m \text{ ch } \tau(\pi-\alpha) \text{ ch } m(\alpha-\theta_0)] dm, \tag{3.12}$$

where $X_1(m) = X(m) + \text{ch } m(\pi-\theta_0)$.

In the derivation of Eqns. (3.11) and (3.12) it was assumed that both $X(m)$ and $Y(m)$ are even functions of m. The assumption is based on the observation that these functions are even in m for the case $\beta_1 = \beta_2$ (Appendix B).

The spectral displacements in each wedge are obtained from Eqns. (3.1) and (3.2)

$$U_2(r, \theta) = \frac{2s_0}{\pi^2} \int_0^\infty Y(\tau) \text{ ch } \tau(\pi-\theta) K_{i\tau}(\beta_2 r) K_{i\tau}(\beta_1 r_0) d\tau, \qquad \alpha \leqq \theta \leqq \pi, \tag{3.13}$$

$$U_1(r, \theta) = \begin{cases} \dfrac{2s_0}{\pi^2} \displaystyle\int_0^\infty X_1(\tau) \text{ ch } \tau\theta \, K_{i\tau}(\beta_1 r) K_{i\tau}(\beta_1 r_0) d\tau, & 0 \leqq \theta \leqq \theta_0 \leqq \alpha, \quad (3.14) \\[2ex] \dfrac{2s_0}{\pi^2} \displaystyle\int_0^\infty [X_1(\tau) \text{ ch } \tau\theta - \text{sh } \tau(\theta-\theta_0) \text{ sh } \pi\tau] K_{i\tau}(\beta_1 r) K_{i\tau}(\beta_1 r_0), & 0 \leqq \theta_0 \leqq \theta \leqq \alpha. \quad (3.15) \end{cases}$$

4. Interaction of a plane wave with a vertical discontinuity

We observe (Appendix B) that for $\beta_1 = \beta_2$ and $\alpha = \frac{1}{2}\pi$ the solution for $Y(m)$ is proportional to ch $m\theta_0$. It is therefore reasonable to put in Eqn. (3.11) the trial solution $Y(m) = A_0$ ch mb for $\alpha = \frac{1}{2}\pi$. The equation for A_0 and b is then

$$A_0 \lambda \text{ ch } \frac{\pi\tau}{2} \int_0^\infty Q(\tau, m) K_{im}(\beta_1 r_0) \left[\text{sh } m\left(\frac{\pi}{2} + b\right) + \text{sh } m\left(\frac{\pi}{2} - b\right)\right] m\, dm$$
$$+ A_0 \tau \text{ sh } \frac{\pi\tau}{2} \int_0^\infty Q(\tau, m) K_{im}(\beta_1 r_0) \left[\text{ch } m\left(\frac{\pi}{2} + b\right) + \text{ch } m\left(\frac{\pi}{2} - b\right)\right] dm$$
$$= \frac{\pi^2}{2} \text{ ch } \tau\theta_0 K_{i\tau}(\beta_1 r_0). \tag{4.1}$$

Executing the integration over m with the aid of Eqn. (2.8) and Table 1, we find with $\text{Re}\{b\} \leqq \frac{1}{2}\pi$, $\rho_0 = \beta_1 r_0 / \beta_2$

$$\tau \text{ sh } \frac{\pi\tau}{2} \int_0^\infty K_{i\tau}(\beta_1 r) \left\{\frac{K_0[\beta_2\sqrt{(r^2 + \rho_0^2 - 2r\rho_0 \sin b)}] + K_0[\beta_2\sqrt{(r^2 + \rho_0^2 + 2r\rho_0 \sin b)}]}{2K_{i\tau}(\beta_1 r_0)}\right\} \frac{dr}{r}$$
$$+ \lambda \text{ ch } \frac{\pi\tau}{2} \frac{\partial}{\partial b} \int_0^\infty K_{i\tau}(\beta_1 r) \left\{\frac{K_0[\beta_2\sqrt{(r^2 + \rho_0^2 - 2r\rho_0 \sin b)}] - K_0[\beta_2\sqrt{(r^2 + \rho_0^2 + 2r\rho_0 \sin b)}]}{2K_{i\tau}(\beta_1 r_0)}\right\} \frac{dr}{r}$$
$$= \frac{\pi}{2A_0} \text{ ch } \tau\theta_0. \tag{4.2}$$

TABLE 1

A selected list of Kontorovich–Lebedev transforms

$F(\tau)=\int_0^\infty f(r)K_{i\tau}(\beta r)\frac{dr}{r},\quad \beta>0$	$f(r)=\frac{2}{\pi^2}\int_0^\infty \tau\,\mathrm{sh}\,\pi\tau\,F(\tau)K_{i\tau}(\beta r)d\tau$
$\frac{\pi\,\mathrm{ch}\,\tau(\pi-\lvert\theta-\theta_0\rvert)}{\tau\,\mathrm{sh}\,\pi\tau}K_{i\tau}(\beta r_0)$ $0<\lvert\theta-\theta_0\rvert\leqq 2\pi$	$K_0[\beta\sqrt{\{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)\}}]$
$2^{\alpha-2}\beta^{-\alpha}\Gamma\left(\frac{\alpha+i\tau}{2}\right)\Gamma\left(\frac{\alpha-i\tau}{2}\right)$	r^α
$\frac{\pi}{2\tau\,\mathrm{sh}\,\pi\frac{\tau}{2}}$	1
$Q(\tau,m)$	$K_{im}(\beta_1 r)$
$\pi^2\frac{\delta(\tau+m)+\delta(\tau-m)}{2\tau\,\mathrm{sh}\,\pi\tau}=Q_0(\tau,m)$	$K_{im}(\beta r)$
$(32\pi\beta r)^{-\frac{1}{2}}\Gamma\left[\frac{1}{4}+\frac{i}{2}(m+\tau)\right]\Gamma\left[\frac{1}{4}-\frac{i}{2}(m+\tau)\right]\times\Gamma\left[\frac{1}{4}+\frac{i}{2}(m-\tau)\right]\Gamma\left[\frac{1}{4}-\frac{i}{2}(m-\tau)\right]$	$K_{im}(\beta r)$
$\frac{1}{r_0}K_{i\tau}(\beta r_0)$	$\delta^+(r-r_0)$
$\frac{\pi}{2\tau\,\mathrm{sh}\,\pi\tau}[K_{i(\tau-m)}(\beta r_0)+K_{i(\tau+m)}(\beta r_0)]$	$K_{im}[\beta(r+r_0)]$
$\frac{\mathrm{ch}\,\tau\Omega}{\tau^2+\lambda^2}K_{i\tau}(\beta r_0)$	$I_\lambda(\beta r_<)K_\lambda(\beta r_>)\cos\lambda\Omega$
$\frac{\alpha}{\tau\,\mathrm{th}(\pi-\alpha)}K_{i\tau}(\beta r_0)$	$\sum_{n=0}^\infty \varepsilon_n I_{\mu n}(\beta r_<)K_{\mu n}(\beta r_>),\quad \mu=\frac{\pi}{\pi-\alpha}$
$\frac{1}{\beta}\frac{\pi}{\mathrm{sh}\,\pi\tau}\frac{\mathrm{sh}\,\Omega\tau}{\sin\Omega},\quad \cos\Omega=\frac{\alpha}{\beta}\cos\eta\leqq 1$	$r\,e^{-\alpha r\cos\eta}$
$\frac{\pi}{\beta\,\mathrm{sh}\,\pi\tau}\frac{\sin\Omega\tau}{\mathrm{sh}\,\Omega},\quad \mathrm{ch}\,\Omega=\frac{\alpha}{\beta}\cos\eta\geqq 1$	
$\frac{\pi}{\tau\,\mathrm{sh}\,\pi\tau}\mathrm{ch}\,\Omega\tau,\quad \cos\Omega=\frac{\alpha}{\beta}\cos\eta\leqq 1$	$e^{-\alpha r\cos\eta}$
$\frac{\pi}{\tau\,\mathrm{sh}\,\pi\tau}\cos\Omega\tau,\quad \mathrm{ch}\,\Omega=\frac{\alpha}{\beta}\cos\eta\geqq 1$	
$b\frac{\mathrm{ch}\,a\tau}{\tau\,\mathrm{sh}\,b\tau}$	$\sum_{n=0}^\infty(-)^n\varepsilon_n I_{(\pi n)/b}(\beta r)\cos\left(\pi n\frac{a}{b}\right),\quad b-a\geqq\frac{\pi}{2}$

As we let $r_0 \to \infty$, the expressions in the curly braces in the two integrands tend to the limits $\operatorname{ch}(\beta_2 r \sin b)$ and $\operatorname{sh}(\beta_2 r \sin b)$ respectively. The remaining integrals are found to be (Table 1)

$$\int_0^\infty K_{i\tau}(\beta_1 r)\operatorname{ch}(\beta_2 r \sin b)\frac{dr}{r} = \frac{\pi}{2}\,\frac{\operatorname{ch}\Omega_2\tau + \operatorname{ch}\Omega_1\tau}{\tau \operatorname{sh}\pi\tau},$$

$$\frac{\partial}{\partial b}\int_0^\infty K_{i\tau}(\beta_1 r)\operatorname{sh}(\beta_2 r \sin b)\frac{dr}{r} = \frac{\pi}{2}\,\frac{\beta_2}{\beta_1}\,\frac{\cos b}{\sin\Omega_2}\,\frac{\operatorname{sh}\Omega_2\tau - \operatorname{sh}\Omega_1\tau}{\operatorname{sh}\pi\tau}, \tag{4.3}$$

where

$$\cos\Omega_1 = -\frac{\beta_2}{\beta_1}\sin b\,,\quad \cos\Omega_2 = \frac{\beta_2}{\beta_1}\sin b\,,\quad \Omega_1 = \Omega_2 + \pi\,,$$

and provided that $\sin b \leqq \beta_1/\beta_2$. Substituting Eqns. (4.3) in Eqn. (4.2) and noting that $\operatorname{sh}\Omega_2\tau - \operatorname{sh}\Omega_1\tau = 2\operatorname{ch}[\frac{1}{2}(\Omega_1+\Omega_2)]\tau \operatorname{sh}\frac{1}{2}\pi\tau$ and $\operatorname{ch}\Omega_2\tau + \operatorname{ch}\Omega_1\tau = 2\operatorname{ch}[\frac{1}{2}(\Omega_1+\Omega_2)]\tau \operatorname{ch}\frac{1}{2}\pi\tau$ we have

$$Y(m) = \frac{2\operatorname{ch} bm}{1 + \lambda\dfrac{v_1}{v_2}\dfrac{\cos b}{\cos\theta_0}} \tag{4.4}$$

$$\sin b = \frac{v_2}{v_1}\sin\theta_0\,. \tag{4.5}$$

The physical situation is described in Fig. 1. A plane wave is impinging from the right upon the formation and part of the energy is transmitted to the left side of the discontinuity. The spectral displacements are given by Eqn. (3.13)

$$\begin{aligned} U_2(r,\omega) &= \lim_{r_0\to\infty,\, s_0\to\infty} \frac{4A_0 s_0}{\pi^2}\int_0^\infty \operatorname{ch} bm \operatorname{ch} m(\pi-\theta) K_{im}(\beta_2 r) K_{im}(\beta_1 r_0)\, dm \\ &= \frac{A_0}{\pi}\lim_{r_0\to\infty,\, s_0\to\infty}[s_0 K_0(\beta_1 r_0)] \\ &\times \frac{K_0[\beta_2\sqrt{\{r^2+\rho_0^2-2r\rho_0\cos(\theta-b)\}}] + K_0[\beta_2\sqrt{\{r^2+\rho_0^2-2r\rho_0\cos(\theta+b)\}}]}{K_0(\beta_1 r_0)}, \end{aligned}$$

$$b \leqq \frac{\pi}{2},\ \theta \leqq \frac{\pi}{2}, \tag{4.6}$$

that is,

$$U_2(r,\omega) = \frac{2}{\pi}\,\frac{\bar{s}_0}{1+\lambda\dfrac{v_1}{v_2}\dfrac{\cos b}{\cos\theta_0}}\cos\left(\frac{\omega r}{v_1}\sin\theta\sin\theta_0\right)\exp\left\{-\mathrm{i}\frac{\omega r}{v_2}|\cos(\pi-\theta)|\cos b\right\}, \tag{4.7}$$

where $\bar{s}_0$ is the renormalized source magnitude. As long as $v_1 \geqq v_2$ or $\sin\theta_0 \leqq v_1/v_2 \leqq 1$, $\cos b$ is real and the wave travels unattenuated. If however $v_1/v_2 < \sin\theta_0 \leqq 1$, b becomes complex:

$$b = \frac{\pi}{2} + \mathrm{i}\chi\,,\quad \begin{matrix}\cos b = -\mathrm{i}\operatorname{sh}\chi \operatorname{sgn}\omega \\ \sin b = \operatorname{ch}\chi\end{matrix}\,,\quad \operatorname{sh}\chi = \sqrt{\left[\left(\frac{v_2}{v_1}\sin\theta_0\right)^2 - 1\right]}\,. \tag{4.8}$$

The spectral field is then governed by the expression

$$U_2(r,\omega) = \frac{1}{\pi}\,\frac{2\bar{s}_0\cos\left(\dfrac{\omega r}{v_1}\sin\theta\sin\theta_0\right)}{1-\mathrm{i}\left[\lambda\dfrac{v_1}{v_2}\dfrac{\operatorname{sh}\chi}{\cos\theta_0}\operatorname{sgn}\omega\right]}\exp\left\{-|\omega|\frac{r}{v_2}|\cos(\pi-\theta)|\operatorname{sh}\chi\right\}. \tag{4.9}$$

The decay factor is proportional to $\exp\{-(|\omega| v_2^{-1}\operatorname{sh}\chi)x\}$ where x is the distance of the observation point from the vertical discontinuity.

The corresponding time-functions are obtained by evaluating the Fourier transforms of Eqns. (4.7) and (4.9),

$$U_2(\boldsymbol{r}, t) = \frac{1}{\pi} \frac{1}{\left[1+\lambda \dfrac{v_1}{v_2} \dfrac{\cos b}{\cos \theta_0}\right]} \left[f(t-t_1)+f(t-t_2)\right], \tag{4.10}$$

$$v_1 \geqq v_2 \text{ or } \sin \theta_0 \leqq \frac{v_1}{v_2} \leqq 1$$

$$U_2(\boldsymbol{r}, t) = \frac{2}{\pi} \frac{\dfrac{v_1}{v_2} \bar{s}_0 \dfrac{\operatorname{sh} \chi}{\cos \theta_0}}{\left[1 + \left(\lambda \dfrac{v_1}{v_2} \dfrac{\operatorname{sh} \chi}{\cos \theta_0}\right)^2\right]} \left[F(t-T_1)+F(t+T_1)\right], \tag{4.11}$$

$$\frac{v_1}{v_2} < \sin \theta_0 \leqq 1$$

where

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{s}_0(\omega) e^{i\omega t} d\omega, \tag{4.12}$$

$$F(t) = \frac{\dfrac{r}{v_1} |\cos(\pi-\theta)| \cos \theta_0 - \lambda t}{t^2 + \dfrac{r^2}{v_2^2} \cos^2 \theta \operatorname{sh}^2 \chi}, \quad T_1 = \frac{r}{v_1} \sin \theta \sin \theta_0, \tag{4.13}$$

$$t_2 = T_1 + \frac{r}{v_2} |\cos(\pi-\theta)| \cos b = \frac{r}{v_2} \cos[b-(\pi-\theta)] > t_1, \tag{4.14}$$

$$t_1 = -T_1 + \frac{r}{v_2} |\cos(\pi-\theta)| \cos b = \frac{r}{v_2} \cos[b+(\pi-\theta)]. \tag{4.15}$$

Note that below the critical angle $\sin \theta_c = v_1/v_2$, the displacements propagate undistorted with the original functional source dependence $f(t)$ [*e.g.* $\delta(t)$ for $\bar{s}_0$=constant]. Beyond the critical angle the signal decays with time and distance. Equation (4.11) is valid only for a delta function time dependence at the source.

The geometrical interpretation of Eqn. (4.10) is shown in Fig. 2. There are two rays which correspond to $f(t-t_1)$ and $f(t-t_2)$. The first arrival is refracted once on the vertical boundary. The path of the late arrival depends on whether $b \lessgtr \pi-\theta$. If $b > \pi-\theta$, the first refraction is at the vertical boundary followed by a reflection at the free surface in medium 2. If $b < \pi-\theta$, the reflection is in medium 1 and it happens prior to the refraction. If $b=\pi-\theta$ i.e. $\sin(\pi-\theta) = v_2 v_1^{-1} \sin \theta_0$, the points of reflection and refraction coincide at 0. For all cases, the time interval between the arrival of the two fronts is one and the same

$$\Delta t = t_2 - t_1 = 2T_1 = \frac{2r}{v_1} \sin \theta \sin \theta_0. \tag{4.16}$$

This is easily verified from geometrical considerations (Fig. 2).

The displacements in medium 1 are obtained in a similar way: we put $X(m) = B_0 \operatorname{ch} am$, $\alpha = \frac{1}{2}\pi$ in Eqn. (3.12), obtaining, after some manipulations

$$2A_0 \int_0^{\infty} \operatorname{ch} am\, Q(m, \tau) K_{im}(\beta_1 r_0) \{2\} dm = \int_0^{\infty} Q(m, \tau) K_{im}(\beta_1 r_0) \{1\} dm, \tag{4.17}$$

where

$$\begin{aligned} \{1\} = \lambda\tau \operatorname{sh} \frac{\pi\tau}{2} &\left[\operatorname{ch} m\left(\frac{3\pi}{2} - \theta_0\right) - \operatorname{ch} m\left(\frac{\pi}{2} - \theta_0\right) - 2 \operatorname{ch} m\left(\frac{\pi}{2} + \theta_0\right)\right] \\ + \operatorname{ch} \frac{\pi\tau}{2} &\left[\operatorname{sh} m\left(\frac{3\pi}{2} - \theta_0\right) + \operatorname{sh} m\left(\frac{\pi}{2} - \theta_0\right) + 2 \operatorname{sh} m\left(\frac{\pi}{2} + \theta_0\right)\right] m, \end{aligned} \tag{4.18}$$

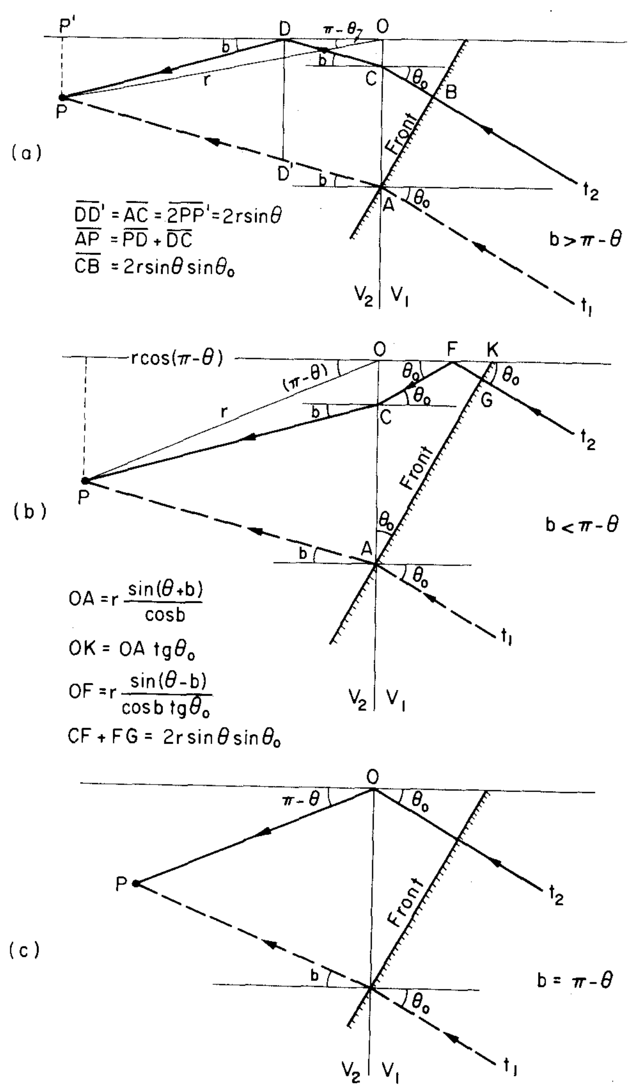


Figure 2. Field of plane-wave behind the discontinuity.

$$\operatorname{ch} am\{2\} = \lambda\tau \operatorname{sh}\frac{\pi\tau}{2}\left[\operatorname{ch} m\left(\frac{\pi}{2}+a\right)+\operatorname{ch} m\left(\frac{\pi}{2}-a\right)\right]$$
$$+m \operatorname{ch}\frac{\pi\tau}{2}\left[\operatorname{sh} m\left(\frac{\pi}{2}+a\right)+\operatorname{sh} m\left(\frac{\pi}{2}-a\right)\right]. \tag{4.19}$$

This equation must render a unique solution for the parameters A_0 and a, valid for all τ. To this end we carry out first the integration over m on both sides of Eqn. (4.17). Using the sum formulae of the hyperbolic functions and the integrals (Table 1)

$$\int_0^\infty Q(m,\tau)K_{im}(\beta_1 r_0)\operatorname{ch} m\eta\, dm = \frac{\pi}{2}\int_0^\infty K_{i\tau}(\beta_2 r)K_0\left[\beta_1\sqrt{(r^2+r_0^2+2rr_0\cos\eta)}\right]\frac{dr}{r}, \tag{4.20}$$

$$\int_0^\infty Q(m,\tau)K_{im}(\beta_1 r_0)\, m \operatorname{sh} m\eta\, dm = \frac{\pi}{2}\frac{\partial}{\partial\eta}\int_0^\infty K_{i\tau}(\beta_2 r)K_0\left[\beta_1\sqrt{(r^2+r_0^2+2rr_0\cos\eta)}\right]\frac{dr}{r};$$

$$\eta \leqq \pi,$$

$$\left.\begin{aligned}&\int_0^\infty K_{i\tau}(\beta_2 r)\,e^{-\beta_1 r\cos\eta}\frac{dr}{r}=\frac{\pi}{\tau\,\mathrm{sh}\,\pi\tau}\,\mathrm{ch}\,\phi\tau\\&\int_0^\infty K_{i\tau}(\beta_2 r)\,e^{-\beta_1 r\cos\eta}\,dr=\frac{\pi}{\beta_2\,\mathrm{sh}\,\pi\tau}\frac{\mathrm{sh}\,\phi\tau}{\sin\phi}\end{aligned}\right\}\begin{aligned}&\cos\phi=\frac{\beta_2}{\beta_1}\cos\eta\\&\beta_2+\beta_1\cos\eta>0\end{aligned}\tag{4.21}$$

together with the limit

$$\lim_{r_0\to\infty}\frac{K_0[\beta_1\sqrt{(r^2+r_0^2+2rr_0\cos\eta)}]}{K_0(\beta_1 r_0)}=\exp[-\beta_1 r\cos\eta]\,,\tag{4.22}$$

we find that Eqn. (4.17) leads to the solution

$$a=\theta_0\,,\qquad B_0=\frac{1-\lambda\dfrac{v_1}{v_2}\dfrac{\cos b}{\cos\theta_0}}{1+\lambda\dfrac{v_1}{v_2}\dfrac{\cos b}{\cos\theta_0}}\,,\tag{4.23}$$

where b is given by Eqn. (4.5). The corresponding displacements are found from Eqns. (3.14) and (3.15)

$$\begin{aligned}&U_1=\frac{2\bar{s}_0}{\pi^2}\int_0^\infty[\mathrm{ch}\,\tau(\pi-\theta_>)+B_0\,\mathrm{ch}\,\tau\theta_>]\,\mathrm{ch}\,\tau\theta_<\,K_{i\tau}(\beta_1 r)K_{i\tau}(\beta_1 r_0)\,d\tau\,,\\&\theta_>=\max(\theta_0,\theta)\,,\quad\theta_<=\min(\theta_0,\theta)\,.\end{aligned}\tag{4.24}$$

The evaluation of Eqn. (4.24) is similar to that of Eqn. (4.6), rendering

$$\begin{aligned}U_1(\boldsymbol{r},\omega)=&\frac{\bar{s}_0}{2\pi}\left[\exp\left\{i\frac{\omega r}{v_1}\cos(\theta-\theta_0)\right\}+\exp\left\{i\frac{\omega r}{v_1}\cos(\theta+\theta_0)\right\}\right]\\&+\frac{\bar{s}_0 B_0}{2\pi}\left[\exp\left\{-i\frac{\omega r}{v_1}\cos(\theta-\theta_0)\right\}+\exp\left\{-i\frac{\omega r}{v_1}\cos(\theta+\theta_0)\right\}\right].\end{aligned}\tag{4.25}$$

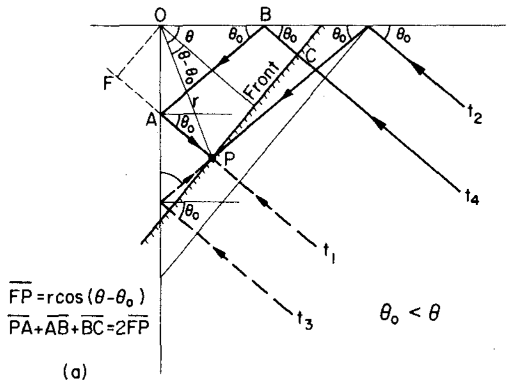
The application of the Fourier transform to (4.25) yields

$$2\pi U_1(\boldsymbol{r},t)=\begin{cases}\left\{f\left[t+\dfrac{r}{v_1}\cos(\theta-\theta_0)\right]+f\left[t+\dfrac{r}{v_1}\cos(\theta+\theta_0)\right]\right\}\\\quad+B_0\left\{f\left[t-\dfrac{r}{v_1}\cos(\theta-\theta_0)\right]+f\left[t-\dfrac{r}{v_1}\cos(\theta+\theta_0)\right]\right\},\\\hfill v_1\geqq v_2\ \text{ or }\ \sin\theta_0\leqq\dfrac{v_1}{v_2}\leqq 1\\\left\{f\left[t+\dfrac{r}{v_1}\cos(\theta-\theta_0)\right]+f\left[t+\dfrac{r}{v_1}\cos(\theta+\theta_0)\right]\right\}\\\quad+\cos 2\varepsilon\left\{f\left[t-\dfrac{r}{v_1}\cos(\theta-\theta_0)\right]+f\left[t-\dfrac{r}{v_1}\cos(\theta+\theta_0)\right]\right\}\\\quad+\sin 2\varepsilon\left\{f_a\left[t-\dfrac{r}{v_1}\cos(\theta-\theta_0)\right]+f_a\left[t-\dfrac{r}{v_1}\cos(\theta+\theta_0)\right]\right\},\\\hfill\dfrac{v_1}{v_2}\leqq\sin\theta_0\leqq 1\,,\end{cases}\tag{4.26}$$

where

$$\mathrm{tg}\,\varepsilon=\lambda\frac{v_1}{v_2}\frac{\mathrm{sh}\,\chi}{\cos\theta_0}\,,\quad B_0=e^{2i\varepsilon}\,,\tag{4.27}$$

and $f_a(t)$, known as the *allied function* of $f(t)$, is defined as



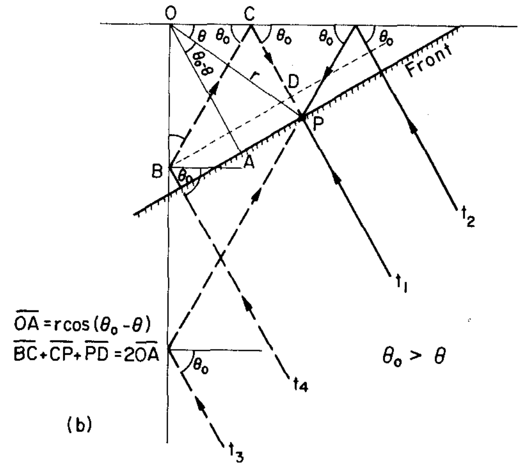


Figure 3. Field of plane-wave in front of the discontinuity.

$$f_a(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{s}_0(\omega)\, e^{[i\omega t + \frac{1}{2}(\pi i)\operatorname{sgn}\omega]}\, d\omega = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\tau)d\tau}{\tau - t} \tag{4.28}$$

[e.g. the allied function of $\delta(t)=\pi^{-1}\int_0^\infty \cos\omega t\, d\omega$ is $\delta_a(t)=\pi^{-1}\int_0^\infty \sin\omega t\, d\omega=(\pi t)^{-1}$ for $t\neq 0$, and $=0$ for $t=0$].

The physical-geometrical interpretation of Eqns. (4.26) is as follows: The field at any point in region 1 is composed of four contributions that arrive at different times. Since the source is at infinity there is no unique fiducial time and the reference time of the solution is the time that the front hits the vertex 0 (Fig. 3). The time it takes the front to move from the observer at (r, θ) to the vertex, in the direction of θ_0, is $t_0 = r v_1^{-1} \cos(\theta - \theta_0)$. The four "rays" arrive therefore at the following times relative to the direct wave,

Direct arrival	$t_1 = 0$,
Single reflection at the free surface	$t_2 = \frac{2r}{v_1}\sin\theta\sin\theta_0$,
Single reflection at the vertical discontinuity	$t_3 = \frac{2r}{v_1}\cos\theta\cos\theta_0$,
Two reflections	$t_4 = \frac{2r}{v_1}\cos(\theta-\theta_0) = t_2 + t_3$.

For angles of incidence, beyond the critical angle $(\theta_0 > \sin^{-1} v_1 v_2^{-1})$ the shape of the pulses reflected from the discontinuity is distorted since each Fourier component is phase-shifted by the amount 2ε.

The total spectrum in region 1 can be rewritten as

$$U_1(\mathbf{r}, \omega) =$$

$$= \frac{\bar{s}_0}{\pi}\cos\left[\frac{\omega r}{v_1}\sin\theta\sin\theta_0\right]\sqrt{\left\{(1+B_0^2)+2B_0\cos\left(\frac{2\omega r}{v_1}\cos\theta\cos\theta_0\right)\right\}}\, e^{i\psi},$$

$$\tan\psi = \left[\frac{1-B_0}{1+B_0}\right]\tan\left(\frac{\omega r}{v_1}\cos\theta\cos\theta_0\right), \tag{4.29}$$

$$v_1 \geqq v_2 \text{ or } \sin\theta_0 \leqq \frac{v_1}{v_2} \leqq 1, \quad B_0 < 1.$$

Comparing Eqns. (4.29) and (4.7) we note that both spectra have "holes" due to *destructive interference* at frequencies f_n which depend both on θ_0 and the coordinates of the observation point

$$f_n = \left(\frac{2n+1}{4}\right) \frac{v_1}{r \sin \theta \sin \theta_0}, \qquad n = 0, 1, 2, \dots . \tag{4.30}$$

Beyond the critical angle, for $\omega > 0$

$$U_1(r, \omega) = \frac{2\bar{s}_0}{\pi} \cos\left[\frac{\omega r}{v_1} \sin \theta \sin \theta_0\right] \cos\left[\frac{\omega r}{v_1} \cos \theta \cos \theta_0 - \varepsilon\right] e^{i\varepsilon}, \tag{4.31}$$

$$U_2(r, \omega) = \frac{2\bar{s}_0 \cos \varepsilon}{\pi} \cos\left[\frac{\omega r}{v_1} \sin \theta \sin \theta_0\right] \exp\left\{-\frac{\omega r}{v_1} |\cos(\pi - \theta)| \operatorname{sh} \chi + i\varepsilon\right\}, \tag{4.32}$$

and the location of the spectral zeros are unchanged. In region 1, however, there is an additional set of holes at

$$f_k = \left[\left(\frac{2k+1}{4}\right) + \frac{\varepsilon}{2\pi}\right] \frac{v_1}{r \cos \theta \cos \theta_0}, \qquad k = 0, 1, 2, \dots . \tag{4.33}$$

The spectral amplitude ratio of U_1 and U_2 below the critical angle, for the same values of r and $\sin \theta$ on both sides of the discontinuity, is found from Eqns. (4.7) and (4.29) to be

$$\frac{U_1}{U_2} = \sqrt{\{1 - \gamma^2 \sin^2(k_1 x \cos \theta_0)\}} < 1, \tag{4.34}$$

where

$$\gamma = \sqrt{(1 - B_0^2)} < 1, \quad k_1 = \frac{\omega}{v_1}, \qquad x = r \cos \theta .$$

The reader can easily verify that Eqns. (4.7), (4.9) and (4.25) indeed satisfy the boundary conditions (3.3).

A short discussion on the field of a line source at a *finite* distance from the vertex is given in Appendix B.

5. Conclusions

The effect of a vertical discontinuity on the propagation of plane waves in an elastic half-space is as follows:

(1) Part of the field is thrown back and made to interfere with the incident radiation. The ensuing wave pattern developes spectral holes at an infinite set of frequencies that depend on the medium parameters as well as upon the coordinates of the observation point (r, θ) and the angle of incidence θ_0 of the plane wave.

(2) It is convenient to compare the field in the presence of the discontinuity to the field in its absence. Their ratio is a measure of the anomaly due to the velocity and rigidity contrasts. Thus, below the critical angle

$$\left|\frac{U_2(\alpha = \frac{1}{2}\pi, \lambda, v_1, v_2)}{U_2(\alpha = \frac{1}{2}\pi, 1, v_1 = v_2)}\right| = \frac{2}{1 + \dfrac{\mu_2}{\mu_1} \dfrac{v_1}{v_2} \dfrac{\cos b}{\cos \theta_0}}, \qquad \theta_0, b \leqq \tfrac{1}{2}\pi, \tag{4.35}$$

$$\left|\frac{U_1(\alpha = \frac{1}{2}\pi, \lambda, v_1, v_2)}{U_1(\alpha = \frac{1}{2}\pi, 1, v_1 = v_2)}\right| = \sqrt{\left\{1 + B_0^2 + 2B_0 \cos\left(\frac{2\omega r}{v_1} \cos \theta \cos \theta_0\right)\right\}} \geqq (1 - B_0). \tag{4.36}$$

The maxima of both ratios is two.

Appendix A

The K–L transform of a function $f(r)$, $0 < r < \infty$ is given by the relation

$$F(\tau) = \int_0^\infty f(r) K_{i\tau}(\beta r) \frac{dr}{r}, \tag{A.1}$$

where τ is real and β is a complex constant. If $f(r)$ is such that $r^{-1}f(r)$ is continuously differentiable and both $rf(r)$ and $r\,d/dr[r^{-1}f(r)]$ are absolutely integrable over the positive real axis, the inversion formula assumes the form [5],

$$f(r) = \frac{2}{\pi^2}\int_0^\infty F(\tau)K_{i\tau}(\beta r)\tau \operatorname{sh} \pi\tau\, d\tau\,. \tag{A.2}$$

This pair of reciprocal formulas can be combined to yield the integral theorem

$$f(r) = \frac{2}{\pi^2}\int_0^\infty \tau \operatorname{sh} \pi\tau\, K_{i\tau}(\beta r)d\tau \int_0^\infty f(\xi)K_{i\tau}(\beta\xi)\frac{d\xi}{\xi}\,. \tag{A.3}$$

Writing Eqn. (2.3) in the form $f(r)=\int_0^\infty f(\xi)\delta^+(r-\xi)d\xi$ where $\delta^+(x)=2H(x)\delta(x)$ is the unit impulse function. $\{\delta(x)$ is the usual Dirac function and $H(x)$ is the Heaviside unit step function$\}$, we obtain the representation

$$\delta^+(r-r_0) = \frac{2}{\pi^2 r_0}\int_0^\infty \tau \operatorname{sh} \pi\tau\, K_{i\tau}(\beta r)K_{i\tau}(\beta r_0)d\tau\,. \tag{A.4}$$

Similarly, from Eqns. (A.1) and (A.2)

$$F(\tau) = \frac{2}{\pi^2}\int_0^\infty K_{i\tau}(\beta r)\frac{dr}{r}\int_0^\infty F(m)K_{im}(\beta r)m \operatorname{sh} \pi m\, dm\,, \tag{A.5}$$

and therefore

$$\delta(\tau+m)+\delta(\tau-m) = \frac{2}{\pi^2}\tau \operatorname{sh} \pi\tau \int_0^\infty K_{i\tau}(\beta r)K_{im}(\beta r)\frac{dr}{r}\,, \quad \beta>0\,. \tag{A.6}$$

Furthermore, since [5]

$$K_{i\tau}(y) = \int_0^\infty e^{-y \operatorname{ch} \xi}\cos(\tau\xi)d\xi\,, \tag{A.7}$$

and

$$\pi\delta(\tau-m) = \int_0^\infty \cos\xi(\tau-m)\,d\xi\,, \tag{A.8}$$

it follows that

$$\pi\delta(\tau-m) = K_{i(\tau-m)}(0)\,. \tag{A.9}$$

Consequently, Eqn. (A.6) can be recast in the form

$$Q_0(\tau, m) = \int_0^\infty K_{i\tau}(\beta r)K_{im}(\beta r)\frac{dr}{r} = \frac{\pi}{2}\frac{K_{i(\tau+m)}(0)+K_{i(\tau-m)}(0)}{\tau \operatorname{sh} \pi\tau}\,. \tag{A.10}$$

The integral $\int_{-\infty}^{\infty} K_{i\tau}(x)d\tau = \pi e^{-x}$ verifies that the normalization constant in (A.9) is correct.

Appendix B

We invoke the Sommerfeld integral [6, p. 20; 7, p. 89] in a slightly modified form

$$\pi H_0^{(2)}(kx) = \int_{\eta-i\infty}^{\mu+i\infty} e^{-ikx\cos(\psi-\zeta)}\,d\psi\,, \tag{B.1}$$

$$\pi<\mu<2\pi\,,\ 0<\eta<\pi\,,\ 0<\zeta<\pi\,,$$

and also the integral representation of the addition theorem for circularly cylindrical waves [8, p. 374]

$$\pi H_0^{(2)}(kR) = \int_{\eta-i\infty}^{\mu+i\infty} e^{ikr\cos\psi - ikr_0\cos(\psi+\theta-\theta_0)}\,d\psi\,, \tag{B.2}$$

where $R^2 = r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)$ (Fig. 1). Then, recalling that the relevant two-dimensional Green's function is $\{-\frac{1}{4}iH_0^{(2)}(kR)\}$, the spectral field due to a line source at (r_0, θ_0) is obtained by a straightforward integration of Eqns. (4.7) and (4.25) over θ_0. In this process we let everywhere in the integrands $\theta_0 \to \psi$ and $\zeta \to \theta_0$. Thus

$$U_1(r/r_0; \omega) = -\frac{s_0 i}{4}\left\{H_0^{(2)}\left[\frac{\omega}{v_1} R\right] + H_0^{(2)}\left[\frac{\omega}{v_1}\hat{R}\right]\right\} - \frac{is_0}{4\pi}\{F(\theta-\theta_0) + F(\theta+\theta_0)\}, \quad \text{(B.3)}$$

$$U_2(r/r_0; \omega) = -\frac{is_0}{4\pi}\{G(\theta_0 + \pi - \theta) + G(\theta_0 - \pi - \theta)\}, \quad \text{(B.4)}$$

where

$$\hat{R} = r^2 + r_0^2 - 2rr_0 \cos(\theta + \theta_0),$$

$$F(\theta) = \int_{\eta - i\infty}^{\mu + i\infty} B_0(\psi) e^{-i(\omega/v_1)[r\cos\psi + r_0\cos(\psi+\theta)]} d\psi, \quad \text{(B.5)}$$

$$G(\theta_0) = \int_{\eta - i\infty}^{\mu + i\infty} [1 + B_0(\psi)] e^{-i(\omega/v_2)[r\cos b(\psi) + r_0\cos(\psi - \theta_0)]} d\psi, \quad \text{(B.6)}$$

$$B_0(\psi) = \frac{\cos\psi - \lambda\frac{v_1}{v_2}\sqrt{\left\{1 - \left(\frac{v_2}{v_1}\sin\psi\right)^2\right\}}}{\cos\psi + \lambda\frac{v_1}{v_2}\sqrt{\left\{1 - \left(\frac{v_2}{v_1}\sin\psi\right)^2\right\}}}, \quad b(\psi) = \sin^{-1}\left[\frac{v_2}{v_1}\sin\psi\right]. \quad \text{(B.7)}$$

The integrals in Eqns. (B.5) and (B.6) can be approximated by the saddle-point method. The case $v_1 = v_2$ lends itself however to exact evaluation. Indeed under the conditions $\mu_1 \neq \mu_2$, $\beta_1 = \beta_2$, the integral expressions in Eqns. (3.11) and (3.12) degenerate into two algebraic equations in $Y(\tau)$ and $X(\tau)$. Their solution renders

$$Y(\tau) = \left[\frac{2}{1+\lambda}\right]\frac{\text{ch}\,\tau\theta_0\,\text{sh}\,\pi\tau}{\text{sh}\,\pi\tau + \sigma\,\text{sh}\,\tau(\pi - 2\alpha)}, \quad \text{(B.8)}$$

$$\sigma = \frac{\lambda - 1}{\lambda + 1} = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} \leqq 1.$$

$$X(\tau) = \left[\frac{1-\lambda}{1+\lambda}\right]\frac{\text{ch}\,\tau\theta_0\,\text{sh}\,2\tau(\pi - \alpha)}{\text{sh}\,\pi\tau + \sigma\,\text{sh}\,\tau(\pi - 2\alpha)}, \quad \text{(B.9)}$$

The spectral displacements in each wedge are then obtained from Eqns. (3.13), (3.14) and (3.15)

$$U_2(r,\theta) = \frac{4s_0}{(1+\lambda)\pi^2}\int_0^\infty \frac{\text{sh}\,\pi\tau\,\text{ch}\,\tau(\pi-\theta)\,\text{ch}\,\tau\theta_0}{\text{sh}\,\pi\tau + \sigma\,\text{sh}\,\tau(\pi - 2\alpha)} K_{i\tau}(\beta r) K_{i\tau}(\beta r_0) d\tau, \quad \text{(B.10)}$$

$$\alpha \leqq \theta \leqq \pi,$$

$$U_1(r,\theta) = \frac{2s_0}{\pi^2}\int_0^\infty \frac{\text{sh}\,\pi\tau\{\text{ch}\,\tau(\pi - \theta_>) - \sigma\,\text{ch}\,\tau(\pi - 2\alpha + \theta_>)\}\,\text{ch}\,\tau\theta_<}{\text{sh}\,\pi\tau + \sigma\,\text{sh}\,\tau(\pi - 2\alpha)}$$
$$\times K_{i\tau}(\beta r) K_{i\tau}(\beta r_0) d\tau, \quad \text{(B.11)}$$
$$0 \leqq \theta_0 \lesseqgtr \theta \leqq \alpha,$$
$$\theta_> = \max(\theta, \theta_0), \quad \theta_< = \min(\theta, \theta_0).$$

The *surface displacements* are obtained from Eqns. (B.10) and (B.11) by the substitution $\theta_< = \theta = 0$, $\theta_> = \theta_0$.

For $\alpha = \frac{1}{2}\pi$, the quadrature of Eqns. (B.3) and (B.4) is immediate:

$$U_2(r,\omega) = \frac{-is_0}{2(1+\lambda)}\left\{H_0^{(2)}\left[\frac{\omega}{v}\sqrt{(r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0))}\right]\right.$$
$$\left. + H_0^{(2)}\left[\frac{\omega}{v}\sqrt{(r^2 + r_0^2 - 2rr_0\cos(\theta + \theta_0))}\right]\right\}, \quad \text{(B.12)}$$

$$U_1(r,\omega) = -\frac{is_0}{4}\left\{H_0\left[\frac{\omega}{v}\sqrt{(r^2+r_0^2-2rr_0\cos(\theta-\theta_0))}\right]\right.$$
$$+\left.H_0^{(2)}\left[\frac{\omega}{v}\sqrt{(r^2+r_0^2-2rr_0\cos(\theta+\theta_0))}\right]\right\}$$
$$+\frac{is_0}{4}\sigma\left\{H_0\left[\frac{\omega}{v}\sqrt{(r^2+r_0^2+2rr_0\cos(\theta-\theta_0))}\right]\right.$$
$$+\left.H_0^{(2)}\left[\frac{\omega}{v}\sqrt{(r^2+r_0^2+2rr_0\cos(\theta+\theta_0))}\right]\right\}.$$

The relation $K_0(i\omega v^{-1}R) = -\frac{1}{2}\pi i H_0^{(2)}(\omega v^{-1}R)$ was used in the derivation of Eqn. (B.8). Note that Eqns. (5.5) and (5.6) reduce to Eqn. (B.8) for $v_1 = v_2$. Assuming a delta-function time dependence at the source, the passage to the time domain is done by means of the hook-integral

$$\frac{1}{2\pi}\oint e^{i\omega t}\left[-\frac{\pi i}{2}H_0^{(2)}\left(\frac{\omega}{v}R\right)\right]d\omega = \frac{H\left(t-\frac{R}{v}\right)}{\sqrt{\left(t^2-\frac{R^2}{v^2}\right)}} \tag{B.13}$$

where $H(t)$ is the unit step-function of Heaviside.

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